NASA TN D-1126

NASA TN D-1126

TECHNICAL NOTE

D-1126

A DEVELOPMENT OF THE LUNAR AND SOLAR DISTURBING FUNCTIONS FOR A CLOSE SATELLITE

William M. Kaula

Goddard Space Flight Center
Greenbelt, Maryland

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

WASHINGTON January 1962

LJ

A DEVELOPMENT OF THE LUNAR AND SOLAR DISTURBING FUNCTIONS FOR A CLOSE SATELLITE

by

William M. Kaula

Goddard Space Flight Center

SUMMARY

The complete gravitational disturbing function

$$R = m^* \sum_{n=2}^{\infty} \frac{r^n}{r^{*\,n+1}} P_n(S)$$

is developed in the form

$$R = m^* \sum_{\substack{n,m,p,\\h,q,j}} \frac{a^n}{a^{*\,n+1}} \kappa_m \frac{(n-m)!}{(n+m)!} F_{nmp}(i)\, F_{nmh}(i^*)\, H_{npq}(e)\, G_{nhj}(e^*)$$

$$\cdot \cos\Big[(n-2p)\omega + (n-2p+q)M - (n-2h)\omega^* - (n-2h+j)M^* + m(\Omega-\Omega^*)\Big]$$

where the functions F, H, G are polynomials. This form is advantageous for conserving computer storage space, making changes in the terms included in the disturbing function, or including the luni-solar perturbations in the same computation with perturbations due to anomalous variations of the earth's gravitational field. A quasi-potential for the solar radiation pressure effect is also developed for use in equations of motion written in terms of Keplerian elements.

TN D-1126

CONTENTS

A DEVELOPMENT OF THE LUNAR AND SOLAR DISTURBING FUNCTIONS FOR A CLOSE SATELLITE

by

William M. Kaula

Goddard Space Flight Center

INTRODUCTION

Mathematical developments of the gravitational effects of the sun and moon on a close satellite have been made by Musen, Bailie, and Upton (Reference 1) and by Kozai (Reference 2). The developments presented herein were made as a result of attempts to adapt the foregoing developments to a form convenient in analyzing close satellite orbits for terrestrial gravitational field variations along the lines of Reference 3. In Reference 3 the disturbing functions are expressed in osculating Keplerian elements for use in equations of motion in those terms (such as in Reference 4, page 147, or Reference 5, page 289).

GRAVITATIONAL DISTURBING FUNCTION

The disturbing function has the form (Reference 1, p. 3)

$$R = m^* \sum_{n=2}^{\infty} \frac{r^n}{r^{*n+1}} P_n(S),$$

where $P_n(S)$ is the Legendre polynomial of S, the cosine of the angle between the position vectors of the satellite and the disturbing body with respect to the earth, and where the parameters of the disturbing body are designated by asterisks (a^*, e^*, etc.).

We define

$$R_n = \frac{m^* r^n}{r^{*n+1}} P_n(S) .$$

And by the addition theorem (from any text on spherical harmonics)

$$R_n = \frac{m^* r^n}{r^{*n+1}} \left[P_n(\sin \delta) P_n(\sin \delta^*) + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} \right.$$

$$\left. \cdot P_{nm}(\sin \delta) P_{nm}(\sin \delta^*)(\cos m\alpha \cos m\alpha^* + \sin m\alpha \sin m\alpha^*) \right] , \qquad (1)$$

where $P_{nm}(\sin \delta)$ is the associated Legendre polynomial of $\sin \delta$. In Equation 1, set

$$A_n^m = \frac{m^*}{r^{*n+1}} \kappa_m \frac{(n-m)!}{(n+m)!} P_{nm}(\sin \delta^*) \cos m\alpha^*$$

and

$$B_n^m = \frac{m^*}{r^{*n+1}} \kappa_m \frac{(n-m)!}{(n+m)!} P_{nm}(\sin \delta^*) \sin m\alpha^* , \qquad (2)$$

where $\kappa_o = 1$; $\kappa_m = 2$, $m \neq 0$.

We apply to Equation 1 the same development as that given in Equations 7 through 18 of Reference 3:

$$R_n = r^n \sum_{m=0}^{n} \sum_{p=0}^{n} F_{nmp}(i) \left\{ \begin{bmatrix} A_n^m \\ \\ -B_n^m \end{bmatrix}_{(n-m)\ \mathrm{odd}}^{(n-m)\ \mathrm{even}} \cos \left[(n-2p)u + m\Omega \right] \right.$$

$$\left. + \begin{bmatrix} B_n^m \\ \\ A_n^m \end{bmatrix}_{(n-m)\ \mathrm{odd}}^{(n-m)\ \mathrm{even}} \sin \left[(n-2p)u + m\Omega \right] \right\} , \qquad (3)$$

where (from Equation 19, Reference 3)

$$F_{nmp}(i) = \sum_t \frac{(2n-2t)!}{t!(n-t)!(n-m-2t)!\,2^{(2n-2t)}} \sin^{(n-m-2t)} i \sum_s \binom{m}{s} \cos^s i$$

$$\cdot \sum_c \binom{n-m-2t+s}{c} \binom{m-s}{p-t-c} (-1)^{(c-k)},$$

in which k is the integral part of $(n-m)/2$; t is summed from 0 to the lesser of p or k; s is summed from 0 to m; and c is summed over all values making the binomial coefficients non-zero.

By applying to Equation 2 the same development as that in Reference 3, we have

$$\left.\begin{aligned} A_n^m &= \frac{m^*}{r^{*3}} \kappa_m \frac{(n-m)!}{(n+m)!} \sum_{h=0}^{n} F_{nmh}(i^*) \begin{bmatrix}\cos\\ \sin\end{bmatrix}_{(n-m)\ \text{odd}}^{(n-m)\ \text{even}} \left[(n-2h)u^* + m\Omega^*\right]; \\ B_n^m &= \frac{m^*}{r^{*3}} \kappa_m \frac{(n-m)!}{(n+m)!} \sum_{h=0}^{n} F_{nmh}(i^*) \begin{bmatrix}\sin\\ -\cos\end{bmatrix}_{(n-m)\ \text{odd}}^{(n-m)\ \text{even}} \left[(n-2h)u^* + m\Omega^*\right]. \end{aligned}\right\} \qquad (4)$$

Combining Equations 3 and 4, we have

$$R_n = \frac{m^* r^n}{r^{*n+1}} \sum_{m=0}^{n} \kappa_m \frac{(n-m)!}{(n+m)!} \sum_{p=0}^{n} F_{nmp}(i) \sum_{h=0}^{n} F_{nmh}(i^*) \Big\{\cos\left[(n-2p)u + m\Omega\right]$$

$$\cdot \cos\left[(n-2h)u^* + m\Omega^*\right] + \sin\left[(n-2p)u + m\Omega\right] \sin\left[(n-2h)u^* + m\Omega^*\right]\Big\}$$

$$= \frac{m^* r^n}{r^{*n+1}} \sum_{m=0}^{n} \kappa_m \frac{(n-m)!}{(n+m)!} \sum_{p=0}^{n} F_{nmp}(i) \sum_{h=0}^{n} F_{nmh}(i^*) \cos\left[(n-2p)u - (n-2h)u^* + m(\Omega - \Omega^*)\right]. \qquad (5)$$

Substituting $n = 2$ in Equation 5 yields the same expressions given as Types I through V, pages 5-6, Reference 1, and $n = 3$ as Types VI through XIII, pages 6-9, Reference 1. Since terms of index $(m,p,h) = (o,p,h)$ will be equal to terms of index $(o, n-p, n-h)$ in Equation 5 above, in R_n there will be a total of $(n+1)^3 - I\left[(n+1)^2/2\right]$ terms, where $I\left[(n+1)^2/2\right]$ is the integral part of $(n+1)^2/2$.

To develop the disturbing function in terms of the mean anomaly of the perturbed body, Hansen's function

$$X_{(n-2p+q)}^{n,(n-2p)}(e)$$

(Reference 4, pages 44-46; Reference 6, page 256) may be used.

The coefficient of $\cos\left[(n-2p)\omega + (n-2p+q)M - (n-2h)u^* + m(\Omega-\Omega^*)\right]$ is

$$a^n H_{npq}(e) = a^n X^{n,(n-2p)}_{(n-2p+q)}(e) . \tag{6}$$

The only terms likely to be of significance are those of long period, $n - 2p + q = 0$. An integration utilizing the true anomaly in a manner similar to Equations 23 and 24 of Reference 3 results in an infinite series which does not converge for large eccentricities. However, Hansen's function reduces to a fairly compact form:

$$a^n H_{np(2p-n)}(e) = a^n X^{n,(n-2p)}_{0}(e)$$

$$= \frac{a^n(-\beta)^{(n-2p')}}{(1+\beta^2)^{(n+1)}}\binom{2n+1-2p'}{n-2p'}\sum_k \binom{n+1}{k}\binom{2p'+1}{k}\frac{\beta^{2k}}{\binom{n-2p'+k}{k}} , \tag{7}$$

where

$$p' = p,\ p \le n/2; \quad p' = n-p,\ p \ge n/2$$

and

$$\beta = \frac{e}{1+\sqrt{1-e^2}} .$$

In terms of the mean anomaly of the disturbing body, for the coefficient of the term $\cos\left[(n-2p)u - (n-2h)\omega^* - (n-2h+j)M^* + m(\Omega-\Omega^*)\right]$ we have

$$\frac{1}{a^{*n+1}} G_{nhj}(e^*) = \frac{1}{a^{*n+1}} X^{-(n+1),(n-2h)}_{(n-2h+j)}(e^*) \tag{8}$$

for the short period terms, of which the form programmed is that given on pages 8-9 of Reference 3 and page 256 of Reference 6; and

$$\frac{1}{a^{*n+1}} G_{nhj}(e^*) = \frac{1}{a^{*n+1}(1-e^{*2})^{n-\frac{1}{2}}}\sum_{d=0}^{h'-1}\binom{n-1}{2d+n-2h'}\binom{2d+n-2h'}{d}\left(\frac{e^*}{2}\right)^{2d+n-2h'} \tag{9}$$

for the long period terms. Combining Equations 5, 6 or 7, and 8 or 9 finally yields

$$R_n = m^* \frac{a^n}{a^{*n+1}} \sum_{m=0}^{n} \kappa_m \frac{(n-m)!}{(n+m)!} \sum_{p=0}^{n} F_{nmp}(i) \sum_{h=0}^{n} F_{nmh}(i^*) \sum_{q=-\infty}^{\infty} H_{npq}(e)$$

$$\cdot \sum_{j=-\infty}^{\infty} G_{nhj}(e^*) \cos \Big[(n-2p)\omega + (n-2p+q)M$$

$$- (n-2h)\omega^* - (n-2h+j)M^* + m(\Omega - \Omega^*)\Big] . \qquad (10)$$

In Equation 10, R_2 is equivalent to Types $a\Omega_1$ through $a\Omega_5$, on pages 12-16 of Reference 1; and R_3 to Types $a\Omega_6$ through $a\Omega_{13}$ on pages 16-24 of Reference 1. For practical purposes only the long period terms $n - 2p + q = 0$ are significant, in which case the summation with respect to q can be omitted and $H_{np(2p-n)}(e)$ taken from Equation 7. Considering only the long period terms,

$$\bar{R}_n = m^* \frac{a^n}{a^{*n+1}} \sum_{m=0}^{n} \kappa_m \frac{(n-m)!}{(n+m)!} \sum_{p=0}^{n} F_{nmp}(i)\; H_{np(2p-n)}(e)$$

$$\cdot \sum_{h=0}^{n} F_{nmh}(i^*) \sum_{j=-\infty}^{\infty} G_{nhj}(e^*) \cos \Big[(n-2p)\omega - (n-2h)\omega^*$$

$$- (n-2h+j)M^* + m(\Omega - \Omega^*)\Big] . \qquad (11)$$

In Equation 11, $\bar{R}_2$ is equivalent to $a[\Omega]_1$ through $a[\Omega]_5$ on pages 25-27 of Reference 1, and to the disturbing function of Reference 2; $\bar{R}_3$ is equivalent to $a[\Omega]_6$ through $a[\Omega]_{13}$ on pages 27-31 of Reference 1. A single term of $\bar{R}_n$ is conveniently abbreviated as:

$$\overline{R_{nmphj}} = \frac{m^* a^n}{a^{*n+1}} \kappa_m \frac{(n-m)!}{(n+m)!} F_{nmp}(i)\; H_{np(2p-n)}(e)\; F_{nmh}(i^*)$$

$$\cdot\; G_{nhj}(e^*)\; T_{nmphj}(\omega, \omega^*, M^*, \Omega, \Omega^*) . \qquad (12)$$

The quantities T'_{nmphj}, $\overline{T_{nmphj}}$, $\overline{T'_{nmphj}}$ can be defined in the same manner as the S'_{nmpq}, etc. on page 10 of Reference 3; i.e., primes denote derivatives with respect to the argument, and overbars integrals with respect to time. The T_{nmphj} functions can be used with $F_{nmp}(i)$, $H_{np(2p-n)}(e)$ and their derivatives, and with $F_{nmh}(i^*)$ and $G_{nhj}(e^*)$, to obtain the variations of the

Keplerian elements in a manner similar to that applied with the harmonic terms of the terrestrial gravitational field in Reference 3. For example, the variation of the node due to a particular term $\overline{R_{nmphj}}$ given by Equation 12 is

$$\Delta\Omega_{nmphj} = \frac{\kappa_m (n-m)!\ m^* F_{nmh}(i^*)\ G_{nhj}(e^*)}{(n+m)!\ a^{*n+1}} \cdot \frac{a^{n-2}\left(\frac{\partial F_{nmp}(i)}{\partial i}\right) H_{np(2p-n)}(e)\ \overline{T_{nmphj}(\omega, \omega^*, M^*, \Omega, \Omega^*)}}{n\sqrt{1-e^2}\ \sin i} . \tag{13}$$

All the significant first order lunar-solar effects thus can be obtained by using the disturbing function given by Equation 11 in the equations of motion (Reference 4, page 147; Reference 5, page 289) and integrating with respect to time. This procedure is obviously advantageous when we desire to conserve programming time or computer storage space, since all the lunar-solar perturbations can be programmed as a single "nest of DO-loops" and the same instructions are used for every term, with only the values of the subscripts n, m, p, h, j changed. It also is convenient to alter the number of terms to be included, since this change can be made by changing the range of values over which the subscripts n, m, p, h, j are to be cycled in their respective "DO-loops." Finally, it is a convenient form to include in the same orbit computation with terrestrial gravitational effects, since the forms of the equations, such as Equation 13, are very similar, and the subroutines required to obtain $F_{nmh}(i^*)$, $G_{nhj}(e^*)$, $F_{nmp}(i)$, $\partial F_{nmp}(i)/\partial i$ are exactly the same as for the terrestrial gravitational effects, while $H_{np(2p-n)}(e)$ and T_{nmphj} and their derivatives are very similar to $G_{np(2p-n)}(e)$ and S_{nmpq} and their derivatives. A subroutine has been written computing the luni-solar secular effects plus all periodic terms of amplitude greater than a minimum specified in the input. This program requires only 943 spaces of core storage, by utilizing subroutines for $F_{nmp}(i)$, $G_{npq}(e)$ and their derivatives which are also required for terrestrial gravitational effects.

RADIATION PRESSURE DISTURBING FUNCTION

If any force affecting an orbit can be expressed as the gradient of a scalar, then this effect can be represented by a disturbing potential R. This is true for radiation pressure if the shadow effect is neglected, since the radiation pressure can be represented with negligible error as the gradient of $R_p = FX$, $F < 0$, where X is the coordinate of the satellite in an earth-centered system with the X axis pointed toward the sun. To obtain R_P in a coordinate system referred to the satellite's own orbit, apply the appropriate rotation matrices to $\mathbf{q} = \{r \cos f,\ r \sin f,\ 0\}^T$:

$$\begin{aligned} R_P &= F \left\{1,0,0\right\} \mathbf{R_3}(\lambda^*)\ \mathbf{R_1}(\epsilon)\ \mathbf{R_3}(-\Omega)\ \mathbf{R_1}(-i)\ \mathbf{R_3}(-\omega)\ \mathbf{q} \\ &= F\mathbf{j}\,\mathbf{R_{SQ}}\,\mathbf{q}\ , \end{aligned} \tag{14}$$

where λ^* is the true longitude of the sun, ϵ is the inclination of the ecliptic, and the rotation matrices $\mathbf{R_i}\,(El.)$ are defined in Reference 3, pages 19-21. By multiplying out Equation 14, combining terms by appropriate transformation of the inclination functions, and integrating with respect to M we obtain the long period part of R_P:

$$\begin{aligned} \bar{R}_P = -F\,\frac{3ae}{2} \Big\{ &+ \cos^2\frac{i}{2} \sin^2\frac{\epsilon}{2} \cos\,(\omega + \Omega + \lambda^*) \\ &+ \cos^2\frac{i}{2} \cos^2\frac{\epsilon}{2} \cos\,(\omega + \Omega - \lambda^*) \\ &+ \sin^2\frac{i}{2} \cos^2\frac{\epsilon}{2} \cos\,(\omega - \Omega + \lambda^*) \\ &+ \sin^2\frac{i}{2} \sin^2\frac{\epsilon}{2} \cos\,(\omega - \Omega - \lambda^*) \\ &- \frac{1}{2} \sin i\ \sin\epsilon\ \cos\,(\omega + \lambda^*) \\ &+ \frac{1}{2} \sin i\ \sin\epsilon\ \cos\,(\omega - \lambda^*) \Big\}\ . \end{aligned} \tag{15}$$

The $\bar{R}_P$ from Equation 15 may be used in the equations of motion (Reference 4, page 147; Reference 5, page 289) to obtain the same results as those in Reference 7.

If the shadow is taken into account, it is impossible to devise a quasi-potential. Such a quasi-potential would have to be constant within the shadow and proportionate to X outside the shadow. Since X at the entry point to the shadow will generally not be equal to X at the exit point, such a potential will unavoidably give rise to a spurious impulse normal to the shadow boundary at either the entry point, exit point, or both. Hence the integration from exit point to entry point to allow for shadow effect must be made after R_p, including short period terms, has been differentiated, i.e.,

$$\frac{1}{2\pi}\int_{M_0}^{M_1}\frac{\partial R_P}{\partial El.}\,dM$$

must be used in the equations of motion, where M_0 and M_1 represent the mean anomaly at exit and re-entry respectively and where $El.$ denotes any one of the orbital elements. For example, for the semi-major axis

$$\frac{da}{dt} = \frac{1}{an\pi}\int_{M_0}^{M_1}\frac{\partial R_P}{\partial M}\,dM = \frac{F\mathbf{j}\mathbf{R}_{SQ}}{an\pi}\int_{M_0}^{M_1}\frac{\partial \mathbf{q}}{\partial M}\,dM$$

$$= \frac{F\mathbf{j}\mathbf{R}_{SQ}}{n\pi}\left\{\begin{matrix}\cos E\\ \sqrt{1-e^2}\ \sin E\\ 0\end{matrix}\right\}_{E_0}^{E_1}, \qquad (16)$$

where E_0 and E_1 are obtained by solution of the quartic equation for the intersection of the shadow by the orbit, as described in References 8 and 9. In the notation of this paper, the quartic to be solved by iteration is

$$X = \mathbf{j}\mathbf{R}_{SQ}\mathbf{q} = -\sqrt{r^2 - a_e^2}$$

or

$$r_{1,1}a(\cos E - e) + r_{1,2}a(1-e^2)^{\frac{1}{2}}\sin E = -\sqrt{a^2(1-e\cos E)^2 - a_e^2}\ ,$$

where $r_{1,1}$ and $r_{1,2}$ are elements of $\mathbf{R}_{SQ}$ and a_e is the radius of the earth.

Since E_0 and E_1 are functions of $\lambda^*, \Omega,$ and ω, their time variation due to change in these angles must be taken into account before Equation 16 is integrated. In view of the intervention of the quartic, a numerical harmonic analysis appears to provide the best method. The program which has been written specifies as input F, the orbital elements at a reference epoch, and the interval for harmonic analysis, and produces as output a Fourier series of any specified number of terms for the variations of the Keplerian elements.

TN D-1126

REFERENCES

1. Musen, P., Bailie, A., and Upton, E., "Development of the Lunar and Solar Perturbations in the Motion of an Artificial Satellite," NASA Technical Note D-494, January 1961

2. Kozai, Y., "On the Effects of the Sun and the Moon upon the Motion of a Close Earth Satellite," Smithsonian Inst. Astrophys. Obs. Spec. Rep. 22: 1959, pp. 7-10

3. Kaula, W. M., "Analysis of Gravitational and Geometric Aspects of Geodetic Utilization of Satellites," NASA Technical Note D-572, March 1961

4. Plummer, H. C., "An Introductory Treatise on Dynamical Astronomy," New York: Dover Publications, 1960

5. Brouwer, D. and Clemence, G. M., "Methods of Celestial Mechanics," New York: Academic Press, 1961

6. Tisserand, F. F., "Traité de Mécanique Céleste," Tome 1, Paris: Gauthier-Villars et Fils, 1889

7. Musen, P., "The Influence of Solar Radiation Pressure on the Motion of an Artificial Satellite," J. Geophys. Res. 65(5): 1391-1396, May 1960

8. Bryant, R. W., "The Effect of Solar Radiation Pressure on the Motion of an Artificial Satellite," NASA Technical Note D-1063, 1961

9. Kozai, Y., "Effects of Solar Radiation Pressure on the Motion of an Artificial Satellite," Smithsonian Inst. Astrophys. Obs. Spec. Rep. 56: 25-33, 1961

APPENDIX A

List of Symbols

a	semi-major axis of satellite orbit
α	right ascension
δ	declination
e	eccentricity
ϵ	inclination of the ecliptic
E	eccentric anomaly
$F_{nmp}(i)$, $F_{nmh}(i^*)$	inclination polynomials, defined by Equation 19, Reference 3 (or by unnumbered equation, Page 3, TN D-1126)
F	radiation pressure acceleration
f	true anomaly
$G_{nhj}(e^*)$	disturbing body eccentricity polynomial, defined by Equations 21, 24, Reference 3 (or Equations 8, 9, TN D-1126)
$H_{npq}(e)$	satellite eccentricity polynomial, defined by Equations 6, 7, TN D-1126
h	disturbing body inclination subscript
i	angle of inclination to equatorial plane
j	disturbing body eccentricity subscript
$\mathbf{j}$	unit vector $\{1,0,0\}$
k	integral part of $(n-m)/2$
κ_m	2 for $m \neq 0$, 1 for $m = 0$
λ^*	true longitude of the sun
M	mean anomaly
m^*	mass of disturbing body

m order subscript, or secondary wave number, of spherical harmonic

n degree subscript, or primary wave number, of spherical harmonic

Ω right ascension of ascending node

ω argument of perigee

p satellite inclination subscript

P_n Legendre polynomial

P_{nm} Legendre associated polynomial

q satellite eccentricity subscript

R disturbing function

R_p radiation pressure quasi-disturbing function

r radial coordinate

$\mathbf{r}$ position vector

S the cosine of the angle between the position vectors of the satellite and the disturbing body with respect to the earth's center

$T_{nmphj} = \cos\left[(n-2p)\omega - (n-2h)\omega^* - (n-2h+j)M^* + m(\Omega - \Omega^*)\right]$: variable part of a term in the disturbing function

u argument, $\omega + f$

X satellite coordinate in a geocentric coordinate system with the X axis pointing toward the sun

$X_c^{ab}(e)$ Hansen's eccentricity function (defined by Equation 21, Reference 3)